Mathematics Education Development
Research in Teaching ← → Learning in Practice

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According to Stenhouse (1984), "research is systematic *inquiry* made public". By bringing *inquiry* into teaching practice we promote learning in three layers: learning of mathematics; learning of the *teaching* of mathematics; and learning of the processes through which mathematics teaching and learning develop. Through examples of developmental practice from school-based research in the UK and Norway, and research in university-based mathematics teaching in the UK, I discuss ways in which mathematics teachers and mathematics educators can form communities of inquiry to promote development in learning and teaching and look critically at some of the issues involved. These issues raise challenges for promoting development within a philosophy of educational practice and at scale, with reference to the wider dimensions of society, system and culture.

Preamble

In tune with the focus of this mathematics education conference, *Curriculum in Focus: Research-Guided Practice*, I have taken *research* and *practice* as two key words, together with key elements of *curriculum*, *teaching* and *learning*. As a *mathematics education* practitioner, I see *mathematics* as central to all of research, practice, teaching and learning.

I started my career as a secondary teacher of mathematics; since then I have been a mathematics teacher at university level, a teacher of teachers in programmes for prospective or practising teachers of mathematics, and a teacher of research students undertaking research into mathematics teaching and learning. In all of this I have been *a teacher*: for me, *teaching matters.* Alongside much of this teaching I have also been a researcher and, as a researcher, I have been a learner. One of the main things I have learned over these years is that developing as a teacher requires one to be a learner, and the overt stance of being a *learner-teacher* is a *research* stance.

A term I have come to use increasingly is that I am a *didactician*. I spent some years working in Norway which had a profound effect on my learning. I worked in a University department called Matematikk Didaktikk – the Didactics of Mathematics – hence I became a didactician. I learned to recognise didactics as different from and complementary with pedagogy. As I understand it (and I am still open to correction) didactics is about the transformation of the subject (mathematics) into activity and tasks through which learners can gain access to mathematics, engage with mathematics, and come to know mathematical concepts. In contrast, pedagogy is about creating the learning environment through which learners' engagement with mathematics can take place effectively. Shulman (e.g., 1986; 1987) and others have referred to an overlap of pedagogy and didactics which they refer to as *pedagogical content knowledge* (PCK). Being a didactician involves developing forms of knowledge such as PCK, knowledge of mathematics, knowledge of theoretical perspectives in learning and teaching mathematics and knowledge in the practice of research. Teachers too have a broad list of forms of knowledge (Ball, Thames & Phelps,

2008; Ponte & Chapman, 2006; Rowland, Huckstep, & Thwaites, 2005) and I believe that collaborating with teachers to promote better learning and teaching of mathematics in classrooms is one of the main tasks and pleasures of being a didactician. Before going further, I will illustrate this work through an example from research in The Teaching Triad Project (TTP), an extension of The Mathematics Teacher Enquiry (MTE) Project undertaken in the UK in the late 1990s.

Example 1 – Collaborating in the use of a theoretical tool

In the TTP, two didacticians and two secondary mathematics teachers (from different schools) formed a small community of inquiry to develop the classroom teaching of mathematics. The teachers had been part of the MTE Project (Jaworski, 1998) in which they had been introduced to a theoretical construct, the Teaching Triad (Characterising teaching in terms of Management of Learning, Sensitivity to Students and Mathematical Challenge – Jaworski, 1994). They wanted to use the triad to explore aspects of their own teaching; we didacticians agreed to support them and used the triad to analyse data from their classrooms (Potari & Jaworski, 2002). Both teachers had very strong views on what they wished to achieve in their teaching: Jeanette was strong on developing students' self esteem, but felt that her mathematical challenge could be strengthened. Sam, an enthusiastic mathematician, knew that for some students his approach could be too challenging, so he wanted to work on being more sensitive to students. Both teachers managed the learning environment to promote development according to their chosen focus. We observed their lessons, talked with them before and after lessons and held many meetings outside school to discuss progress and issues. The issues raised were not always straightforward or easy to deal with for either teachers or didacticians, but both groups learned from the experience of working together in these ways. Didacticians needed to draw on theory to rationalise their findings (Jaworski & Potari, 2009).

Inquiry and research

I described our collaboration in the TTP as a community of inquiry. The teachers were inquiring into aspects of their own practice in order to develop practice. The didacticians were inquiring into their use of the triad and whether the triad could be used to analyse developing practice. In a sense, all of us were engaging in research. For the teachers it was a form of action research in their own teaching. For the didacticians it was more formal research in which we gathered and analysed data against research questions in a rigorous way. Stenhouse (1984) has said that "research is systematic inquiry made public". In writing about our inquiry we make it public, and there are other forms of dissemination, for example teachers talking with teachers about what they do.

In the MTE project, which had involved 6 teachers from 5 schools, teachers had felt uncomfortable calling themselves *researchers*, so that talking about *inquiry*, rather than *research*, helped them to feel more comfortable with undertaking developments in their practice (Jaworski, 1998). Talking together, about their aims for developing practice, made inquiry in practice more possible. For some teachers, inquiry took the form of using inquiry-based mathematical tasks with students to encourage students' engagement with mathematics and their associated mathematical understanding. Thus, they engaged with inquiry in two layers: inquiry in mathematics with students in the classroom; inquiry into teaching as they reflected on what happened when they put their planning into practice and

analysed its outcomes. The didacticians working with these teachers had a dual role: 1) supporting the teachers in their inquiry activity and 2) charting development in the project. In some cases the first of these raised dilemmas which became a subject of the second. For example when one teacher became 'stuck' and did not know how to continue, and seemed in danger of dropping out of the project, didacticians had to decide whether to suggest to her how she might continue, or whether to leave her to grapple with her dilemma. In the event, they did the latter and the teacher did sort out her own issues (Jaworski, 1998), but they took a risk and the result could well have been different. From such events, didacticians learned about the developmental research process – we might say they engaged in *critical alignment*, as I shall discuss shortly. In a later study, with colleagues in Norway, didacticians recognised over and over again, the *critical* dimension of an inquiry community in which asking critical questions about our own practice while engaging in that practice is an important if not always comfortable learning process (see also, Goodchild, 2014).

Inquiry in the curriculum

The use of inquiry-based tasks to promote students' engagement with mathematics might be seen as the beginnings of an inquiry curriculum. Gueudet and Trouche (2011a; b) talk about teachers' 'documentational genesis' in which teachers' knowledge of teaching grows through their use of a range of resources, material and intellectual, and their 'schemes of utilisation' of the resources. We might see inquiry-based teaching, to some extent, as the use of inquiry-based tasks and the schemes of utilisation of these tasks. The design of the task is itself an important part of the teaching process, and the schemes of utilisation are an important part of the design. As I have suggested above, central to inquiry is the asking of questions, particularly critical questions which challenge the status quo and make us start to think differently. Finding answers to the critical questions would be ideal, but it is rather the addressing of such questions which allows for a change in practice. Cochran-Smith and Lytle (1999) have talked about 'inquiry as stance', through which teachers take on the role of inquirers in their practice; similarly I have talked of inquiry as a 'way of being' in which inquiry becomes central to how we act – constantly questioning, asking why we do things the way we do them (Jaworski, 2004a). Developing such a stance, or way of being in the classroom enables a teacher to work effectively with inquiry-based tasks, to develop effective schemes of utilisation. I think another example is needed.

Example 2 – An inquiry-based task and its schemes of utilisation

In our work with teachers from 8 schools ranging from lower primary to upper secondary in the Learning Communities in Mathematics (LCM) Project in Norway, didacticians designed inquiry-based tasks for use with teachers in university workshops and some teachers adapted these tasks for activity in their classrooms. Didacticians designed the tasks drawing on their own experience and a range of resources. One task we called the mirror task was posed as follows: How tall a mirror must you buy if you want to be able to see your full vertical image? Because we were working with teachers from a range of schools, we wanted such a task to be usable at all levels from Grade 1 to Grade 13. We therefore presented a scenario for the use of the task at each grade level as follows.

Grade 1 – One student faces a mirror holding a stick (against his/her stomach). This student directs another, who, using a whiteboard marker, marks the mirror image the first one sees. Compare the original stick with the marks on the mirror. Try different distances from the mirror.

Grade 2 – One student holds a geometric figure (against the stomach) and explains to another student how to draw (on the mirror) the mirror image he sees. Compare.

Grade 3 – Measure yourself in centimetres. Measure your mirror image in centimetres. Draw yourself seeing yourself in a mirror.

Grade 4 – Have a mirror with a grid. One student holds a geometrical figure (against
stomach) and explains how another student can draw this on the mirror. Count number of squares (area) and compare.

Grade 7 – Draw model of a figure and an eye and the mirror image the eye sees (keep the eye and the figure at the same distance from the mirror?). Describe lengths and angles. What do you see?

Grade 8 – Hold a cube and go close to the mirror. Draw on the lines of the cube on the mirror. What do you see?

Grade 11 – How tall a mirror must you buy if you want to be able to see your full vertical image? Justify your conclusion; try with objects with different distances from mirror; describe ratios in your model

Grade 12 – How tall a mirror must you buy if you want to be able to see your full vertical image? Justify your conclusion; try with objects with different distances from mirror; describe ratios in your model use the cosine rule to derive the height of the actual figure when the height of the mirror image is known

Grade 13 – Draw yourself and a mirror in a three dimensional vector space.

These scenarios offered the basis of schemes of utilisation of the mirror task with students. Teachers needed to adapt these ideas for their own use. One teacher Trude adapted the task for her Grade 9 students. She reported on her students' activity in our next workshop. She had given her students small mirrors and columns of cubes as had been used in the original workshop. However, she had not made clear to them how they might use these resources and so the activity had not seemed to result in the understanding she had sought (Jaworski, Goodchild, Eriksen, & Daland, 2011). Thus, the scheme of utilization employed had not been sufficient to the task and the outcome had not been what was desired. As she explained in her presentation to colleagues at a subsequent workshop, she learned a lot from the exploratory process.

This example illustrates the relationship between a task and its schemes of utilisation and demonstrates the many considerations that come into play when we design inquiry-based activity for students and look for particular outcomes. An inquiry-based curriculum has to enable teachers to develop their own schemes of utilisation and grow into inquiry ways of being with critical questioning through which they can design and adapt their designs according to students' needs – a challenging demand!

Critical alignment through inquiry

In the above I have talked about the centrality of critical questions within an inquiry community. I should now like to ground some of these terms. The idea of inquiry community is becoming widely used; however, it is not always well defined. Wells (1999) has described inquiry as "a stance towards experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). He draws attention to the multiple facets, qualities and uses of inquiry as an "approach to education" in which it "gives full recognition to the mutually constitutive relationship between individual and society" (p.121). While rooted historically in culture, practices and artefacts, inquiry brings an element of action, through which members of a community develop agency, as well as providing an affective dimension – the inspiration from sharing with others and the excitement of developing new ideas, new knowledge.

Wells suggests that an inquiry community is "a particular type of "community of practice" [CoP] as this concept has been developed by Lave and Wenger (1991)" (p. 122). I have found the ubiquitous theory of *community of practice* especially valuable in rooting inquiry theoretically. As Wells has pointed out, inquiry is essentially a communicative process in which mutual activity leads to action, agency and development of new ways of being and thinking. Such mutuality of experience forms the basis of *community of practice* as proposed by Wenger (1998), building on the earlier work of Lave and Wenger (1991). Wenger talks of three dimensions of practice: *mutual engagement* – establishing norms, expectations, ways of working and social relationships; *joint enterprise* – developing common understandings of what the enterprise is about and where it is going, its aims and ideals; and *shared repertoire* – the objects that we use and how we use them, resources such as technology, symbols, abstract forms. The words "mutual", "joint" and "shared" together emphasise the communicative nature of a community of practice.

Wenger (1998) talks of *learning* as "a process of becoming" (p. 215). This, he claims, is "an experience of identity" (p. 215), where identity "serves as a pivot between the social and the individual, so that each can be talked about in terms of the other" (p. 145). He offers again three dimensions which he calls "modes of belonging" in which *identity* is conceptualised in terms of "belonging" to a *CoP* involving "engagement", "imagination" and "alignment" (p. 173). An individual *engages* with practice, alongside co-practitioners, uses *imagination* to weave a personal trajectory within the practice and *aligns* with the norms and expectations of the practice. Thus individual identity is defined in relation to the individual's participation in practice.

So, imagine a classroom situation with teacher and students working together on mathematical tasks. It is possible to conceptualise the classroom as a community of practice in which the practice is teaching-learning mathematics. Students (we hope) are learning mathematics (alongside other aspects and forms of learning); they might also be teaching each other as they communicate in working together on the tasks. The teacher is teaching the students, and is also learning as a teacher. The nature of learning and teaching in this classroom (and any other) is very complex. We can discuss what is being learned (which often goes beyond the mathematics which is in focus) and we can negotiate the meaning of teaching. What does it mean to teach?

The teaching-learning practice in which students and teacher engage has elements of *mutual engagement*; *joint enterprise*; and *shared repertoire.* I will illustrate this claim in my next example. Through *engaging* and using *imagination*, the participants fulfil their roles in the CoP and develop *identity* with respect to the CoP, *aligning* with the norms and expectations of such a classroom within a school. However, their alignment does not necessarily result in the most effective learning outcomes from the joint activity. Brown and McIntyre (1993) talk of classrooms settling down to the "normal desirable state" in which teachers and students work together comfortably: students are mainly well behaved so long as the teacher does not challenge them too much. Walter Doyle and colleagues (e.g., Doyle, 1988, pp. 173/4) wrote about the kinds of tasks presented in mathematics classrooms and their demand on students. *Familiar tasks*, based primarily in memory, formulas, search and match strategies, have routinised recurring exercises and outcomes that are predictable. *Novel Tasks* require higher cognitive processes, understanding and transfer, and decisions about how to use knowledge; have low predictability and high emotional demands. When *familiar* work is being done, the flow of classroom activity is typically smooth and well ordered. Tasks are initiated easily and quickly, work

involvement and productivity are typically high, and most students are able to complete tasks successfully. When *novel* work is being done, activity flow is slow and bumpy. Rates for student errors and non-completion of work are high. Students negotiate directly with teachers to increase explicitness of product specifications or reduce strictness of grading standards. Tasks which appear to elicit comprehension or analytical skills are often subverted to become routine or algorithmic – the 'desirable state'.

The point I am making here is that a classroom characterised as a community of teaching-learning practice in mathematics is not necessarily the high achieving classroom that educators, parents and politicians would like to see, even though it might be comfortable for teachers and students to work without disruption or challenge. Bringing inquiry to such a classroom challenges the status quo, through critical alignment, with the possibility for greater achievement. However, this process is risky; students may not work willingly in the light of the challenge and the teacher may not have the skills to enable students to work with inquiry-based tasks (as we saw with Trude in the example above).

In terms of Wenger's three elements of identity, engagement, imagination and alignment, *critical alignment* implies a change to the third, *alignment*. Rather than (just) aligning with the norms and expectations of the community of practice, criticial alignment involves looking critically at our practice as we engage in and with it. It involves questioning what we do, asking ourselves "why?" and considering whether we have other alternatives. It does not mean changing everything. So we align with the norms of practice (it is hard to do otherwise) but we bring a questioning perspective and a willingness to innovate, to consider what might be valuable changes and to explore possibilities through an inquiry process. As observed it involves clear elements of risk and these need to be undertaken through a supportive and like-minded inquiry community. In Example 2, the teacher Trude engaged with critical alignment in bringing the mirror task to her students. Although the outcome did not have her envisaged success, she learned from trying out the task in her classroom in a spirit of inquiry: she was able to bring new knowledge to her understandings of practice as expressed to her colleagues in the workshop within our community of inquiry. The next example illustrates this notion of inquiry community.

Example 3: An inquiry-based lesson supported in an inquiry community

This lesson was video-recorded as part of a research project into the use of investigational activities in mathematics lessons to promote students' mathematical engagement (Jaworski, 1994). I worked with a group of teachers in the mathematics department of one school. The foundation year curriculum in mathematics included many investigative or inquiry-based tasks agreed between the teachers in the department. Here, the teacher, George, had chosen a task with which he invited his students (aged 12-13) to engage. The class was seated around tables in which students worked together in friendship groups. Design of teaching involves a didactic process in which the abstract ideas of mathematics are (re)conceptualised by the teacher into mathematical tasks and activity for students. I suggest that the didactic goals demonstrated in this lesson included the following:

- to provide opportunity for students to engage with the topic (in this case, area and perimeter of plane shapes);
- to stimulate language patterns and imagery to contribute to understanding;

- to provide a *need* to practice and apply procedures – not just practice for its own sake;
- to promote students' own exploration and inquiry for motivation and purposeful engagement.

The chosen task was "Four square perimeter". It was stated simply as follows:

What perimeters can we get with four squares placed edge to edge or corner to corner, but not overlapping? (5? 6? 10? 99?)

Two examples of legitimate arrangement of the four squares can be seen in Figure 1.

Figure 1: Ways of arranging four squares

The class had worked on the task for four squares, trying out different arrangements; they had moved on to consider larger numbers of squares as suggested in the question. In all cases, according to the rules of arrangement, the perimeter they found was an even number. This resulted in a conjecture, "the perimeter will always be even", and led to a question, posed by teacher George, "is it possible to find an odd perimeter?"

I focus now on an episode, involving three girls' approach to tackling this question, which was recorded on video. In discussion with the teacher they had suggested that, rather than lining the squares up with full sides touching, they might consider the situation with half squares touching. The teacher encouraged them to explore this. They talked about it and drew various diagrams, and then one girl offered the diagram in Figure 2 and started to count its sides.

Figure 2: Four squares with sides half touching

The other girls joined in the counting. They counted sides and counted again. It seemed to add up to 13. They then counted systematically together – all the whole sides first (there were 10) and then the half sides (6), so altogether 13 sides – a perimeter of 13. One of the girls said, "So you can. If you take half squares you can get an odd number", and the other two nodded in agreement. We then see the teacher return to this group and the girls eager to tell him what they had found. The girls spoke all at once "you can … ", "if you add the half squares …", "you can get an odd number". The teacher looked at their diagram and started to count: one, two, two-and-a-half, …". "No" said the girls, "No, No, count like *this*", and they demonstrated their systematic form of counting. The teacher followed their instructions; he counted 10 whole sides, wrote down 10; he counted the half sides, wrote down 3, then he wrote 13, and said "Hey!". The "Hey" seemed to acknowledge their success. They were all smiling and seemed pleased with themselves. We can see elements here of Wenger's *community of practice*: mutual engagement (teacher and students engaging with the task together); joint enterprise (they have a common question to address

and their exploration leads towards an answer); shared repertoire (their ways of working in groups, exploratory activity, specialising, generalising and conjecturing).

I have described this episode in detail to acknowledge certain aspects or qualities of this lesson. The girls were fully engaged in their investigation. Of course this may have had something to do with their being video-recorded, but nevertheless, there was an unforced spontaneity in their words and actions. They wanted to be sure of what they were finding: I draw this conclusion from the ways in which they drew and re-drew their figures and checked and rechecked their counting. They bounced ideas off each other through half-formed sentences. When the teacher returned to them, they were insistent that he should do the counting in their way – telling him clearly what to do. This demonstrated a confidence in their finding that an odd number was indeed possible with this kind of arrangement.

In terms of what was achieved in this lesson, we might say that these students *knew* perimeter – that perimeter had been 'reified' (it had become a manageable object whose properties were seen as central to it) (Sfard, 1994; Wenger, 1998). They could count it, talk about it, work with it and manipulate it. They showed evidence of mathematical thinking in inquiry activity: of trying out special cases, making conjectures and moving towards generalisation (e.g., Mason, Burton, & Stacey, 1982). They worked well together within a group, built on each other's suggestions, and looked critically at what they had found. We could argue that all they had found was one special case. In the video extract, we did not see them check other numbers of squares. However, their systematic mode of counting could be seen as generic. We might believe they could have applied this to any number of squares. The teacher did not push them to check further. In fact they had answered the question, "is an odd number of squares possible?" The answer was "yes".

With hindsight, it could have been valuable to push them further to address whether this arrangement would reveal an odd perimeter for *any* number of squares and then towards a proof. In fact, an odd perimeter only arises when the number of squares is *even* which might have been revealed with further exploration. However, this is just speculation.

Critical alignment can be seen here in the mathematical process in which the whole class conjecture, that the perimeter must be an even number, was challenged and explored further. It led to a new way of seeing the rules in operation, and to finding at least one odd perimeter. Such activity, albeit at an elementary level, is characteristic of that of research mathematicians and enables students to see mathematics in construction rather than as a set of procedures to use according to given rules. I suggest that this episode demonstrates clear characteristics of learning within an inquiry community for the students.

I was challenged by a colleague who, after reading this text, wrote

> I think it could be argued that this is not necessarily an example of 'critical alignment' if an inquiry stance towards mathematics is one of the accepted norms of the classroom (to which teacher and students are aligned). Is the research mathematician critically aligned to mathematics, or is she/he aligned to the practice of mathematics research? (Simon Goodchild; personal communication)

If I am arguing that critical alignment is a norm of inquiry-based practice (as seen in this classroom), then can alignment to this norm be 'critical'? This is a conundrum! It begs the question: what is the practice to which alignment is critical? In terms of classroom mathematics, the practice (for students) might be seen as completing the tasks set by the teacher; here we see students challenging the teacher as a result of their critical inquiry, thus acting against the teacher's autonomy. In comparing this with the activity of research mathematicians, I need to ask whose autonomy they are satisfying. As they fulfil the norms of mathematics research, itself an inquiry process, progress might require a leap of faith in

quite alternative directions in order to reach a research goal. Simon Singh's account of Andrew Wiles's work on Fermat's Last Theorem seems to fit this proposal (Singh, 1997). I welcome further comments on these ideas.

I now come to the learning of the teacher.

Developing an inquiry identity as a student, teacher or researcher

Wells' (1999) statement that inquiry is "a stance towards experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121), can be applied to the students in the extract in Example 3. These students engaged in inquiry according to Wells's definition. Moreover their overall stance, encouraged by their teacher was one of critical inquiry or critical alignment, as I have argued above. The task was 'novel' in Doyle's terms. They enjoyed their engagement, witnessed by their excited talk and animated faces, which we see clearly in the video. The task was challenging at an appropriate level. They could take up the challenge and work with it. They were confident enough to instruct their teacher. Their relationship with the teacher was one of respect and trust, so that they could take initiative and expect him to listen to them. In terms of the teaching triad, the teacher's mathematical challenge for the students matched his sensitivity to them, in both affective and cognitive terms. The video shows the ease of relationship and ways in which the teacher and students interacted with friendly respect. We can say that sensitivity and challenge were in harmony thus leading to effective educational outcomes (Potari & Jaworski, 2002).

In a video-recorded interview between the researcher, the teacher George and some of his colleagues, after they had viewed the extract together, the teacher said, "I was ad-libbing – I didn't know what would happen for half squares … these girls were teaching me something". George had been prepared to challenge his students, encourage their exploration and listen to their explanation of what they had found. This demonstrates inquiry in his teaching and his learning from the process. So, it appeared that, in encouraging the girls to explore further with the half squares, the teacher was on unknown ground, but willing to take a risk (critical alignment?); perhaps, later, in using this task with other students he would be more aware of possibilities and able to judge whether to push them. He certainly seemed to have learned from the girls' activity and reasoning. Thus we might say that he was in the process, overtly, of developing his teaching. We might even say that he was acting in an inquiry mode in trying out possibilities in his classroom and learning from outcomes. We might be less positive and say that he was taking too many risks and that without the requisite knowledge he might not advise or support his students in the best possible way.

Such issues arose in the discussion of the teachers and researcher. While respectful of George's activity and decisions as a teacher, the other teachers probed teaching decisions and outcomes. One issue, raised by one of George's colleagues, was a challenge to friendship groups, suggesting they might be too "comfortable" and possibly not challenging enough. This was debated, with this teacher and George choosing to disagree. It was clear from this conversation with the teachers that significant issues relating to learning and teaching in mathematics were raised and addressed. As researcher, I kept in the background, but offered comments occasionally when it seemed to me that another perspective might be valuable. In my listening and interaction I was a learner with the teachers. The results of this study pointed overwhelmingly to this mutual learning with

developmental potential and outcomes (Jaworski, 1994). Much of my subsequent work in studying and promoting development practice in mathematics teaching derives from these findings.

Through this example, I seek to highlight ways in which inquiry in practice leads to knowing and doing with confidence, in mathematics learning and teaching. The students explored confidently, demonstrating mathematical processes of specializing, conjecturing and reaching for generality (Mason et al., 1982) evidenced through the video data. We might expect that their success with inquiry in this case, building on earlier similar experiences, contributed to their building an inquiry identity – an inquiry *stance*, or *way of being* (Cochran-Smith & Lytle 1999; Jaworski, 2004b; Wells, 1999). This can be true also for the teachers, bringing inquiry into their practice and analyzing its details and issues together with colleagues and a researcher. It is certainly true for the researcher who gained a wealth of insight from working with these teachers. I have kept in touch with some of these teachers over the years since this research was completed; I believe we all would agree that this was a very fruitful time of learning.

In justifying the activity of these teachers as forming a community of inquiry, I point to the following characteristics, having in mind the work of Wells, 1999, cited above.

- Evidence of "a stance towards experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" in the mathematical activity, exploration of teaching and the research inquiry process.
- Evidence of the multiple facets, qualities and uses of inquiry as an "approach to education" with clear instances of "mutually constitutive relationship between individual and society" – the society here being the school environment, the mathematics inquiry-based curriculum, and the committed team of mathematics teachers.
- Evidence of teacher-researcher collaboration through inquiry leading to development in thinking and practice (Jaworski, 2008a).

Although I present what I consider to be clear evidence of these aspects of critical alignment and community of inquiry, there is always the question of whether this is an isolated example, whether these are special teachers, or whether the enthusiasms and promptings of the researcher are the main ingredients of the practices observed. While this is undoubtedly a special case in a number of ways, it also illustrates what is possible, to which others can aspire. I ask in what ways such practices can be seen more generally or promoted more widely. One example is in the Mathematics Teacher Enquiry project discussed above. Another is in the projects in Norway which I have also mentioned briefly. I return to these projects now as an example of how the ideas of inquiry-based practice can be promoted more widely.

Example 4 – Inquiry-based practice on a larger scale

The first of the Norwegian projects was Learning Communities in Mathematics (LCM), and it ran in parallel with the project ICT in Mathematics Learning (ICTML), (see Jaworski, Fuglestad, Bjuland, Breigteig, Goodchild, & Grevholm, 2007). LCM recruited initially 7 schools from lower primary to upper secondary. ICTML worked also with the secondary schools, and brought another secondary school to the joint projects; thus LCM worked with 8 schools. The basis for both projects was that didacticians from the university

would work with selected teachers from the schools (at least 3 from each school) to develop opportunities for students' learning of mathematics through inquiry-based approaches. Didacticians were mathematics educators including several doctoral students and some teacher-educators (the number varied from 12 to 15 at different times). In all, throughout the project (4 years) we worked with over 30 teachers. The project had three phases, each of one school year focusing on community building, innovation and inquiry, and goal setting. There were two major parts to each phase: (1) workshops in the university for all participants, led by the didacticians (6 per year) and (2) school-based activity led by the teacher team in each school. The latter varied according to the level and particular aims of the school. Three didacticians were associated with each school team; they were willing to engage at school level through invitation from the teachers; liaison between school teams and didacticians was undertaken by doctoral students.

Typically, in a workshop, didacticians and teachers worked together on mathematical tasks mainly prepared by didacticians. "Worked together" is not a euphemism for didactician-led problem-solving; didacticians tried hard to come 'fresh' to these problems and to work with teachers as problem-solvers, not as teachers of teachers (Cestari, Daland, Eriksen, & Jaworski, 2005). Problem solving led to discussion about activity with students in classrooms and issues of pedagogy and didactics were addressed. I have given the example above of the mirror task and Trude's response to it (Example 2).

For the teachers, although the ideas of exploration or investigation in classroom tasks were not new (the mathematics curriculum encouraged such practices) the meaning of "inquiry" was not clear. We worked together in both English and Norwegian, and the English word "inquiry" was used. It became clear that there was no exact Norwegian word to translate "inquiry", so this triggered a wide discussion about what exactly we mean by this term. We found the discussion extremely valuable because it enabled people to try to make their own sense in relation to our discussion. One teacher seemed extremely taken by the ideas generated which he expressed as follows:

> Egil: … the "inquiry" – it is nice to have that concept. ... Before, I had an unreflected attitude to what I was doing, but I did it because I felt it was good ... [Now we are more attuned to being in a process], and the process I think has been very confirming in respect of what you were doing as a teacher … There is one thing you said that for me is the most important thing. That was at the meeting we had in December, when the school leaders were there, and we had a discussion about this inquiry, and when you [a didactician] said that [described] inquiry as *a way of being*. And that just put everything in place [for me]. (Bjuland & Jaworski, 2009, p. 36)

For the teachers, implementation of inquiry-based ideas took on different forms. Teachers in primary and lower secondary education seemed most able to incorporate inquiry-based tasks into their practice in differing ways. There were lots of examples, many of them video-recorded by didacticians at teachers' invitation, such that we produced an extensive video compilation so that teachers could get insights to their colleagues practices and share ideas and issues. The teachers who were most resistant, and quite critical, of ideas to promote inquiry-based practice were at the higher secondary level. Some felt that their curriculum was too demanding for exploratory work which would take too much time and which did not address the particular needs of their students. There were suggestions that didacticians were not interested enough in their issues and not helpful enough with activity in their classrooms (Goodchild, 2014; Goodchild & Jaworski, 2005; Jaworski & Goodchild, 2006). In contrast, however, it was one team of upper secondary teachers who were the first to invite didacticians to help them plan some lessons on linear equations,

with an inquiry base, for their students. This proved extremely fruitful in a range of ways, and led to a very positive presentation from the teachers at a subsequent workshop. Nevertheless they felt that such activity should be used sparingly because of the time taken both to plan the tasks and in the classroom (Goodchild, Fuglestad, & Jaworski, 2013).

Promoting inquiry-based teaching on a larger scale

Publications from the LCM project, and projects which led from it, provide considerable evidence of the value of such collaboration between teachers and didacticians for the knowledge, understanding and practice of both groups and for developing classroom practice. Although the teachers involved were largely self-selecting, they were not 'special' in any other way; all were 'normal' teachers in regular schools. In terms of the debate on critical alignment, initiated above, both teachers and didacticians can be seen to engage with critical alignment with respect to their own established practices. In coming together to form a putative inquiry community (in the project) they undertook to engage in inquiry in their practice. Each group had to learn from the other: teachers had to modify their practices in classrooms to incorporate inquiry-based tasks; didacticians had to modify their designs for workshops to accommodate to teacher's expectations and requests. So the inquiry community involved a growth of understanding as activity was mediated by the challenges from the other group.

These projects were well-funded[1] with a substantial didactician team and enthusiastic young researchers. This of course was 'special'. It allowed us to work in depth with a number of schools and teachers. Nevertheless, these projects could not really be considered to be "at scale". Hennessey (2014) comments on similar collaborative projects taking place in the UK with extremely fruitful outcomes through collaboration between teachers and university researchers although again, not at scale:

> The question of 'scaling up' such endeavours is, of course, a thorny one. We hope our professional development materials, created as a result of this project, will go some way to achieving this ambitious aim. (p. 281).

The suggestion here is that materials developed as part of the projects will enable the work of the projects to have a wider influence than for just those institutions involved in the project. However, it remains to be considered how such collaborations can be conducted on a larger scale as for example, at *district* level (e.g., Cobb & Jackson, 2011; Jackson & Cobb, 2013) or at *national* level (e.g., Krainer 2014; Krainer & Zetemeier, 2013). Krainer describes how the IMST Projects (Innovations in Mathematics, Science and Technology Teaching) were set up nationally in Austria as a result of poor results in TIMSS and PISA testing. These projects were promoted by the government and included both national networks and teacher-led innovation projects. Teacher educators were involved in supporting teachers in inquiry-based activity.

> One of the project's basic interventions is to promote teachers' investigation into their own work. It is assumed that this supports the teachers' critical stance towards innovation and inquiry, which in turn is an important basis for disseminating inquiry-based learning (Krainer & Zetermeier, 2013, p. 875).

Thus these projects seem to have had much in common with the Norwegian projects. Krainer emphasises that both *internal* and *external* resources and support are needed to sustain such activity over more than ten years, and it seems therefore that long term government support and funding were critical to the success of these projects at a national scale. Jackson and Cobb (2013) report from activity at district level in the US, where a federal system of educational devolution means that professional development systems are district-based. Nevertheless, a district covers wide areas and large numbers of schools and a key aspect of the projects involved work with district leaders as well as leadership in schools. Jackson and Cobb write:

> [W]e conducted annual cycles of data collection, analysis and feedback in which we collected and analysed data to document how the districts' improvement strategies were playing out in schools and classrooms; shared our findings with district leaders; and made actionable recommendations about how their improvement strategies might be revised to make them more effective. (Jackson & Cobb, 2013, p. 84)

The 'improvement strategies' included inquiry-based activity with teachers. As with Krainer's projects these district-based projects had both internal and external support – the internal support providing for professional development activity with teachers and the external support providing a super-structure for the dimension of scale. These two examples suggest that teaching development involving inquiry-based practice needs to be sustained over lengthy time periods and therefore needs substantial support from external sources to enable development at scale.

Inquiry-based teaching and a mathematics curriculum

In what I have written above, teaching, learning, tasks, activity, practice and so on, all involve mathematics and relate to a national curriculum for mathematics. The UK National Curriculum (NC) was introduced in 1988. Some of the research quoted above pre-dated this (e.g., Jaworski, 1994). Before the introduction of the NC, teachers were free to choose and use what kinds of tasks they wished, and a lot depended on the perspectives of the particular school, and very much on the national examinations at the time (e.g., Love, 1988). The early version of the NC included a major theme on "Learning and Doing Mathematics" which positively encouraged exploratory mathematics in classrooms. Associated national examinations included a "coursework" element, assessed by teachers, in which students undertook extended (exploratory) tasks in mathematics. In the decades which followed, the NC became more prescriptive in the kinds of tasks recommended and the nature of classroom activity; coursework gradually disappeared, regular formal testing and final examinations became the sole forms of assessment. Within this demanding regime, it was hard for teachers to stray from the prescribed routes and research shows students finding their mathematics programme tedious and unhelpful (Nardi & Steward, 2003). However, very positively, the National Centre for Excellence in Teaching Mathematics [NCETM] was established which conducted developmental and research programmes. One of these "Researching Effective CPD in Mathematics Education (RECME) was set up to

> ... provide advice, guidance and recommendations for the NCETM , in order to inform future plans and to point to the types of evidence that could demonstrate that continuing professional development (CPD) is informing teachers' practice and students' learning. (NCETM, 2009, p. 1).

While the research showed factors that contributed to these aims, such as leadership, challenge, time and networking all building teachers' confidence and contributing to changes in teachers' attitudes towards their teaching, it was shown that teachers had difficulty in gaining evidence of students' actual learning from the initiatives undertaken. The latter raises challenges for the research community.

In Norway, at the time of the LCM research, the National Curriculum in place was the *Laereplanverket* 1997, commonly known as *L97*. Where mathematics was concerned, this curriculum had been written largely by didacticians (Kirke- utdannings- og forsknings-departementet, 1997) who had encouraged investigative work in mathematics classrooms. Nevertheless, a research study at the time showed some teachers not attending to the L97 curriculum document, but rather continuing to teach as they always had: "Curricula come, curricula go – classroom practice endures" (Kleve, 2007, p. 316) For such teachers, it seemed that curricular philosophy had little effect. It seemed that without some form of developmental programme to back up the curriculum, the curriculum itself could not achieve its own aims. The LCM and subsequent projects might be seen to provide such developmental support.

One recognised problem with LCM was that it was designed and promoted almost entirely by didacticians. Schools who volunteered to be in the programme did so through an open invitation from the university. The direct follow-on from LCM was a programme TBM/LBM – Teaching Better Mathematics/Learning Better Mathematics. These were in fact two (joint) programmes. TBM was a (research) programme initiated by didacticians to follow LCM, with funding from the Research Council of Norway. LBM was a (developmental) programme initiated by School Owners and Leaders with funding from the Department of Education. Thus, the combined programme was a joint initiative involving developmental research between didacticians and school personnel, (perhaps) a more democratic enterprise than in LCM. A particular feature of the programme showed teachers from LCM taking on leadership roles in the new programmes, a significant indication of developmental outcomes of this work.

These remarks beg the question of the relationship between a mathematics curriculum and 'effective' mathematics teaching and learning in schools. The word *effective* is hugely controversial because it depends on the particular philosophy underpinning school and classroom activity. In both the UK and Norway, partially in reaction to international testing through TIMSS and PISA (Mullis, Martin, Gonzalez, & Chrostowski, 2004; Organisation for Economic Co-operation and Development, 2001), standardised tests at a variety of levels were imposed widely (through political decree) and teachers felt obliged to conform to the testing regime (teaching to the test through drill and rote learning – e.g., Minards, 2012) to avoid their students being disadvantaged. Thus, 'testing' brought its own educational philosophy which was at odds with a philosophy of inquiry-based practice. This challenges us to think hard about the philosophy behind such practices and how it can be rationalised with the systemic realities of political decision-making and educational organisation. These challenges are axial to those of scale, above; we might imagine the crude image of a 3-dimensional array, with *inquiry-based mathematical practice* being set against the other two axes, and situations represented as points in 3-D space.
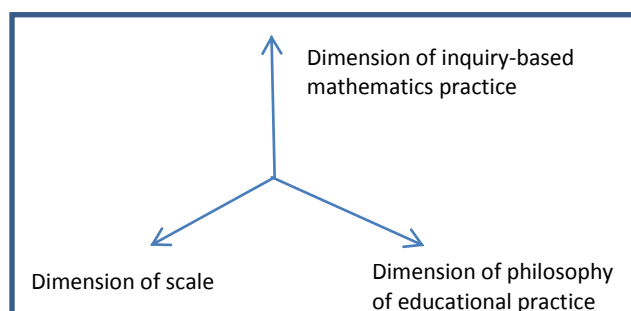
Teaching mathematics at higher levels

In the sections above, I have been addressing issues in the teaching and learning of mathematics at school level and particularly collaborations between teachers and didacticians. I move now to consider teaching mathematics in higher education. This brings with it a range of other considerations and another set of issues. There is a considerable literature relating to mathematics learning at undergraduate level, largely from individual cognitivist perspectives, and a growing literature written by mathematics practitioners about their own teaching. The latter is rarely research based; it consists mainly of personal experience and theories of what works with students. There is a dearth of research that explores mathematics teaching practices in depth (e.g. Artigue, Batanero, & Kent, 2007; Speer, Smith, & Horvath, 2010). Working now closely with a university mathematics department, and teaching mathematics again for the first time for more than 20 years, it is therefore a challenge for me to think about the natures and practices of teaching at this level and what it might mean to bring inquiry approaches to such teaching. In fact, I have been in a position to explore my own teaching, while at the same time finding ways to explore teaching with my new colleagues. Hence I could try to form inquiry communities to explore mathematics teaching. This has involved a steep learning curve with the following findings, some from research others from reflecting on experience:

1. mathematics dominates teaching and there is far less emphasis on didactics and pedagogy than at lower levels;
2. students have more responsibility for their own learning; this influences what teachers offer and what teaching looks like;
3. as with the school system, the university system imposes hugely on philosophy and culture of teaching and learning;
4. innovatory practice (for example inquiry-based teaching) is individually possible and encouraged but hard to promote more widely.

Many of the teachers of mathematics in a university are mathematicians – people who do research in mathematics (e.g., Burton, 2004). Unsurprisingly their focus is on the mathematics they teach. It was a privilege to work closely with one mathematician in researching the teaching of linear algebra and to gain insights into ways in which didactics and pedagogy emerge through considerations of mathematical epistemology. For example, concepts of *subspace*, *linear transformation* and *basis* are central to meaning-making in linear algebra. It was fascinating to look in depth into the ways in which the teacher constructed examples and offered tasks to these students to enable their conceptualisations (Jaworski, Treffert-Thomas, & Bartsch, 2011; Thomas, 2012. See also Nardi, 2008). This research constituted a form of inquiry into the nature of teaching which had 'knock-on' effects for development in teaching, since the teacher modified both his thinking and practice during the research (Jaworski, 2003).

We hold a seminar series entitled 'How we Teach' in which one teacher of mathematics (mathematician or mathematics educator) presents some aspect(s) of their teaching followed by discussion of ideas and issues. Its initial purpose was to *generate* a discourse in mathematics teaching; in practice it seemed to *reveal* such discourse. As we talked about what we do and why, very different ways of conceptualising teaching were revealed. Twenty of these seminars were video-recorded and a sample (of ten) analysed focusing on discourse, revealing a range of findings (Jaworski & Matthews, 2011). For example, some teachers think it is good for students to take their own notes and make sense of them after a lecture; others produce notes and put them onto the VLE (Virtual Learning Environment); yet others produce notes with gaps – students have to attend the lecture to fill in the gaps with solutions in examples and problems. Some lecturers think it vital for students to attend lectures and hence do what they can to encourage attendance; others recognise that many students will not attend, and so they do their utmost to support these students with notes and other materials on the VLE. Very few are overtly reflective on the mathematical meanings students make in relation to the ways they teach.

Here I see a major difference with teaching at lower levels – it is hard to link teaching and learning except through the formal assessment system. For example, in the linear algebra research, with more than 200 students in the cohort and observation of lectures over one semester, it was almost impossible to gain access to students' meaning-making in relation to the observed teaching. It would have been necessary to have regular access to the thinking of individuals or small groups of students which was beyond our resource for the project. In my final example, I address issues in seeking to relate teaching and learning in an inquiry based innovation in first year mathematics teaching.

Example 5 – Innovation in teaching mathematics to engineering students

For several years, I was fortunate enough to teach an introductory mathematics module to a cohort of around 50 first year materials engineering students. I found that, although many had A level mathematics, their understanding of functions was at an instrumental or procedural rather than a relational level (Skemp, 1976) – they saw a function as a rule linking values of x to values of y and were able to distinguish particular functions such as polynomials, trigonometric functions and so on and recognise certain graphical representations such as the parabola, or exponential curve. However, there seemed to be little sense of the nature of the function as an object, in terms of relationships between representations, domain and range, the uniqueness of image, or the existence of an inverse.

With my third cohort of students, I decided to introduce the software GeoGebra to enable a wide range of functions to be explored alongside inquiry-based questions. However, results were disappointing – I needed to be clearer as to the pedagogical approach that paralleled my didactical design (Jaworski, 2008b), and so, I was challenged to undertake a more systematic study. This was done in the following year, in collaboration with two colleagues and a research officer[2]. It involved a design research approach in which inquiry-based mathematical tasks were designed, the cohort was split into small groups, and a small group assessed project was introduced into the assessment structure of the module. Tasks introduced in lectures (using GeoGebra as a demonstration tool) were continued in tutorials in which student groups engaged in exploration using GeoGebra and

[2] Made possible with funding from our UK HE STEM programme – see case-studies – follow links from http://www.thelep.org.uk/national/usefullinks.

followed this up in their own time to complete a group project. Data were collected through observation of lectures and tutorials, surveys of the students, lecturer reflection, written projects of the students, assessment outcomes, and post-module interviews with students. The results were interesting and challenging.

- There was considerable evidence of students' engagement with positive learning outcomes regarding the nature of *function* and relationships between representations, although not all students engaged to the same degree with an inquiry approach.
- Written work from group projects provided evidence that students had understood the purpose of using GeoGebra alongside inquiry based tasks to generate a deeper understanding of functions.
- Examination results were on average higher than with previous cohorts.

However, the interviews produced a different story. The module structure had included a final examination in formal traditional style, as with previous enactments of the module. Questions on the examination had not been inquiry-based questions or used GeoGebra. Students indicated the influence of the examination on their overall engagement with the module and revealed perspectives much in line with school-based practices (as mentioned above, e.g, Minards, 2012). For example, they felt that they had not been given enough opportunity to practice questions from past examination papers; they stated that although GeoGebra might have helped them to understand functions better, it had not helped them with the examination. One said:

> I found GeoGebra almost detrimental because it is akin to getting the question and then looking at the answer in the back of the book. I find I can understand the graph better if I take some values for x and some values for y, plot it, work it out then I understand it … if you just type in some numbers and get a graph then you don't really see where it came from.

It seemed clear that this student had not engaged with GeoGebra in the inquiry style envisaged by the designers of the tasks and their utilisation schemes. These findings from the research generated considerable reflection for the research team. We undertook an activity theory analysis of our findings, juxtaposing the emergent student 'world view' with that of the teaching-research team (Jaworski, Robinson, Matthew, & Croft, 2012). This showed us that cultural, contextual and systemic issues affected strongly the perceptions of those engaged with the module – the teaching team and the student cohort – in terms of what was possible for each in relation to the other and to the wider aspects of the university system. It showed that inquiry-based teaching, albeit well designed and implemented, could not by itself achieve the desired forms of practice and outcome. Systemic and cultural issues were overwhelming in their influence.

Here again we see that inquiry does not provide a magical way of satisfying teaching and learning goals. However, this project had a number of positive outcomes related to its inquiry approach. Critical alignment can be seen in the teaching team's seeking to implement an inquiry curriculum within the systemic constraints, and learning considerably about challenges and needs for change (for example, the powerful influence of maintaining a traditional style examination).

Reflective summary and enduring issues

Key elements of the discussion above are inquiry community, critical alignment and teaching development, which all have their significance in seeking to offer students the best possible environments for learning mathematics. Inquiry is seen as a process which

enables learning at three levels: in classroom mathematics, in mathematics teaching, and in developmental research and practice relating to mathematics teaching. Research, of any sort, is a form of inquiry and results in the creation of knowledge; it includes teacher(-researcher)s inquiring into their own practice, often in informal ways, and (university-)researchers (didacticians) engaging in (more formal) research into the whole developmental process. The brackets in the last sentence indicate the fluidity of roles. What has become evident, and is shown in the growing literature on the topic (e.g., Jaworski & Huang, 2014), is that collaborations between teachers and didacticians can be extremely fruitful in the developmental process.

Inquiry communities are formed when groups of students and/or teachers and/or didacticians work together using inquiry. They can be seen to have a common base with 'communities of practice' (Wenger, 1998) with one major difference concerning *critical alignment*. Examples above show teacher-student, teacher-teacher and teacher-didactician communities. In my own work, *teacher-didactician* communities have focused particularly on creating *teacher-student* communities in classroom mathematics to enable students' mathematical achievement. The creation of inquiry-based mathematical tasks has been seen as central to this work, with research into the use of such tasks in relation to their designed 'schemes of utilisation' a major aspect of developmental research.

Critical alignment is a concept developing from Wenger's notion of 'identity' involving three domains, engagement, imagination, and *alignment*. In a community of inquiry *alignment* becomes *critical alignment*. Rather than just aligning with norms and expectations of the community, participants question their engagement, asking *why* they do things in accepted ways, questioning the outcomes of those ways, and seeking alternatives where outcomes are not as desired. A question which has arisen above concerns what happens when an inquiry community becomes established – whether aligning with the practices of this community is, of itself, critical alignment. This is a question for further consideration.

By *teaching development*, I mean *mathematics* teaching development. In all of the above, I have been talking about the teaching and learning of *mathematics*. It is a pity that sometimes, in talking about teaching development, it is as if the critical element of mathematics somehow gets lost. The inquiry processes on which teaching development is based are very specific to mathematics, and become manifested in the design of mathematical tasks and their schemes of utilisation – examples above are the *mirror task* and the *4-square perimeter task*. Each of these was designed to engage students with mathematics. In the case of *4-square perimeter,* the scheme of utilisation related to the particular group of students for whom the task was designed. In the case of the *mirror task*, a range of schemes of utilisation were suggested relating to students at a range of levels. These would have needed to be fleshed out for the use of the task with particular students. One of the problems experienced by teacher Trude was that her scheme of utilisation was not adequate for her desired outcomes for students.

However, ultimately, teaching development arises from addressing questions about mathematics teaching, how best to approach the mathematics which we desire students to learn and understand. Inquiry-based tasks are one response to such questions. Critical alignment takes us into considerations which go beyond the task itself, and its schemes of utilisation, to *approaches* to teaching and how we design teaching to address complex areas of mathematics (such as 'basis' in linear algebra); the exposition and examples we use, and ways in which we guide students' own engagement and study patterns are all a

part of this. As I engage with my colleagues teaching mathematics in the university, the systemic constraints become central to considering inquiry ways of operation, and it is all too easy to align with established practices which fit these constraints. Critical alignment here becomes problematic when established practices are accepted by both teachers and students even though, for many students, mathematical understanding is limited.

Finally, I come back to the question of *scale*, particularly with regard to teaching students at primary and secondary levels. The projects mentioned above have operated on small or medium scale. In order to operate at large scale, district, region or country, I believe there have to be systemic and cultural influences; for example, as in the collaboration with the school system leaders in TBM (above), in the district work reported by Cobb and colleagues, or the country-wide project reported by Krainer. This takes us into political dimensions, influencing our political leaders, but also into issues of culture which are even harder to address. School and university cultures are embedded in social systems and societal ways of being. In the LCM and TBM projects, we had some local success in both systemic and cultural respects, but these were in local pockets rather than having a national impact. There are big questions to address with regard to wider influences, and this is an enduring challenge for critical alignment. The diagram tries to capture relationships which beg further investigation.
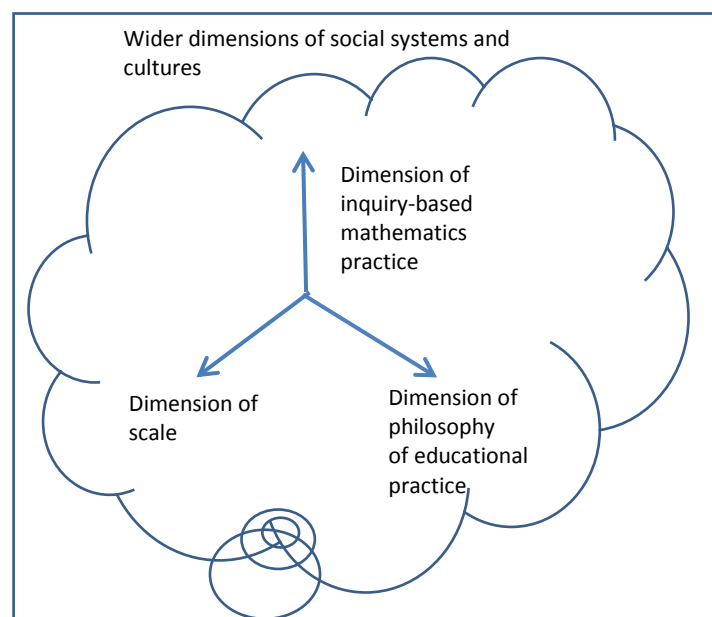
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